Measurement uncertainty relations

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The talk is based on the following papers on the topic in question together with Paul Busch (York) and Reinhard Werner (Hannover):

- *Proof of Heisenberg's Error-Disturbance Relation*, PRL **111** (2013)160405 (2013) [5 pages], arXiv:1306.1565 [quant-ph];
- *Heisenberg uncertainty for qubit measurements*, arXiv:1311.0837 [quant-ph];
- *Measurement uncertainty relations* (forthcoming),
  - a detailed mathematical investigation;
- *Noise operators and measures of rms error and disturbance in quantum mechanics* (forthcoming),
  - a detailed conceptual investigation.

Motivation

Heisenberg's 1927 intuitive ideas with a semiclassical analysis of the γ-ray thought experiment led him to the following conclusion

a position measurement of an electron with an accuracy (resolution of the microscope) q_1 necessarily disturbs its momentum by an amount p_1 such that

$$q_1 p_1 \sim h.$$

Recently:

Scientific American, March 8, 2012:
experimenters violate Heisenberg's original version of the famous maxim, but confirm a newer, clearer formulation.

phys.org, Sep 07, 2012:
scientists cast doubt on renowned uncertainty principle.

Tähdet ja avaruus **5/2013**:
Hiukkaset heittäytyvät kurittomiksi. Vuonna 1927 Werner Heisenberg muotoili kuuluisan epätarkkuusperiaatteen. Nyt se on saatu rikottua laboratoriokokeissa.

More recently:

IN FOCUS NEWS 420 | NATURE | VOL 498 | 27 JUNE 2013
Proof mooted for quantum uncertainty.

Physics Synopsis OCT 17, 2013 :
Rescuing Heisenberg

Physicsworld.com NOV 1, 2013:
Uncertainty reigns over Heisenberg's measurement analogy

Claim: **modern quantum mechanics confirms Heisenberg's intuitions**.

Structure of the talk

Framework - the theory

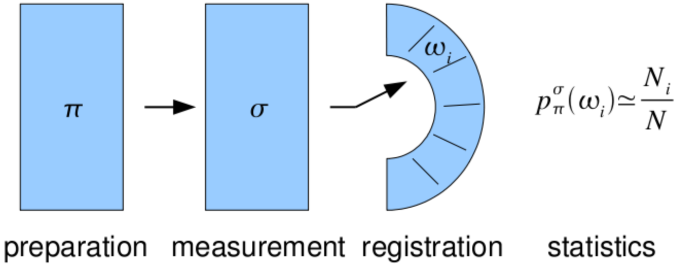
Uncertainty relations

Quantifying errors

The results

What is wrong with the NO-approaches

Frame - statistical causality



state = equivalence class $[\pi]$ of preparations;
observable = equivalence class $[\sigma]$ of measurements;

Assumption: the map $[\pi] \mapsto p^{[\sigma]}_{[\pi]}$ preserves the statistical mixing of preparations π and thus of states $[\pi]$.

Frame - notations/structures

Hilbert space quantum mechanics (with the **Assumption**):

- States $[\pi]$ as density operators ρ (positive trace one ops);
  Then
- Observables $[\sigma]$ as POMs $\mathsf{A} : X \mapsto \mathsf{A}(X)$, with value spaces $(\Omega, \mathcal{A})$, typically $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$;
  An observable is *sharp* if it is projection valued;
- The Born rule: $X \mapsto p_\rho^{\mathsf{A}}(X) = \mathrm{tr}\,[\rho\mathsf{A}(X)] = \mathsf{A}_\rho(X)$;

- Observables A are equivalence classes of (CP) instruments $\mathcal{I}$, instruments $\mathcal{I}$ are equivalence classes of measurements $\mathcal{M} = (\mathcal{K}, \sigma, \mathsf{Z}, U)$:

$$\mathrm{tr}\,[\rho\mathsf{A}(X)] = \mathrm{tr}\,[\mathcal{I}(X)(\rho)] = \mathrm{tr}\,[U\rho \otimes \sigma U^* \mathbf{1} \otimes \mathsf{Z}(X)]$$

Frame - joint measurements

- A measurement $\mathcal{M}$ of A followed by a measurement of B defines a *sequential biobservable* E,

$$\begin{aligned} \mathsf{E}(X,Y) &= \mathcal{I}(X)^*(\mathsf{B}(Y)) \\ \mathsf{E}_1(X) &= \mathsf{E}(X,\Omega_2) = \mathcal{I}(X)^*(\mathsf{B}(\Omega_2)) = \mathsf{A}(X), \\ \mathsf{E}_2(Y) &= \mathsf{E}(\Omega_1,Y) = \mathcal{I}(\Omega_1)^*(\mathsf{B}(Y)) = \mathsf{B}'(Y). \end{aligned}$$

- Such an E extends to a *joint observable* G with

$$G(X \times Y) = \mathsf{E}(X,Y).$$

- If one of the marginal observables is sharp, then the marginal observables commute and E / G are of the product form:

$$\mathsf{E}(X,Y) = \mathsf{E}_1(X)\mathsf{E}_2(Y) = \mathsf{G}(X \times Y) = \mathsf{G}_1(X)\mathsf{G}_2(Y) = \mathsf{A}(X)\mathsf{B}'(Y)$$

An important corollary

Since (sharp) position Q and momentum P are maximal quatities we have the following **FUNDAMENTAL COROLLARY**:

Corollary

A measurement of position Q*, with an instrument* $\mathcal{I}$*, destroys all the momentum information coded in the initial state* ρ *since all the effects* $\mathsf{P}'(Y) = \mathcal{I}(\mathbb{R})^*(\mathsf{P}(Y))$ *of the disturbed momentum* P' *are functions of the position operator Q, that is, all the characteristic properties of momentum (esp. translation invariance) are lost in the measurement, and* vice versa*.*

$$\begin{array}{rcl} \rho & \mapsto & \mathcal{I}(\mathbb{R})(\rho) \\ \mathsf{P}_\rho(Y) & \mapsto & \mathsf{P}_{\mathcal{I}(\mathbb{R})(\rho)}(Y) = \mathsf{P}'_\rho(Y) \end{array}$$

No measurement (weak or not) and **no** definition of error and disturbance can avoid this result.

This also suggests that if there is no error, say, in position measurement, then the disturbance in momentum must be huge.

Uncertainty relations
– three sorts –

One may distinguish between three types of uncertainty relations:

1. For preparations (Kennard-Weyl-Robertson – the text book versions);

2. For the disturbance on (the statistics of) one observable caused by a measurement of another observable;

3. For joint accuracies in approximate joint (or bi-) measurements.

Depending on a pair of observables such relations may look quite different, be of a product form or of a sum form or something else.

2. Error - disturbance relations

In measuring an observable A with a measurement $\mathcal{M}$, the actually measured observable A′ may differ from A.

An operational quantification $\Delta(\mathsf{A}', \mathsf{A})$ of the difference between A′ and A is the *error or accuracy* in measuring A with $\mathcal{M}$.

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The measurement $\mathcal{M}$ causes a change (disturbance) in any other observable B, the disturbed observable B′ being uniquely determined from B by $\mathcal{M}$: $\mathsf{B}'(Y) = \mathcal{I}(\Omega)^*(\mathsf{B}(Y))$.

An operational quantification $\Delta(\mathsf{B}', \mathsf{B})$ of the difference between B′ and B is the *disturbance* of B caused by $\mathcal{M}$.

For a given pair (A, B) the product $\Delta(\mathsf{A}', \mathsf{A}) \cdot \Delta(\mathsf{B}', \mathsf{B})$ may have a strictly positive lower bound for any $\mathcal{M}$.

Such a relation - if exists - is an UR for *error-disturbance*.

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Strictly speaking, and "uncertainty relation" is any inequality that excludes the origin $\Delta(A', A) = 0 = \Delta(B', B)$ and some region around it. For instance,

$$\Delta(A', A)^2 + \Delta(B', B)^2 \geq c > 0.$$

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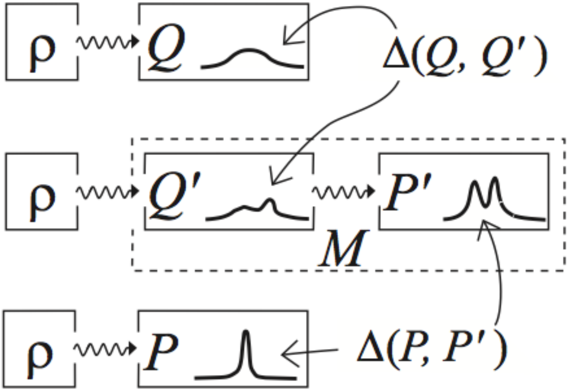
We show that such relations hold for the canonical pair (Q, P) as well as for pairs of qubit observables like s_x, s_y, s_z.

The scenario discussed by Heisenberg for $q_1 \cdot p_1 \sim h$.

The middle row shows an approximate position measurement Q' followed by a momentum measurement.



3. Approximate joint measurements

The error-disturbance scenario is a special case of the approximate joint measurement scenario (since biobservables extend to joint observables).

If A and B are such that they cannot be measured jointly, there is the possibility that they can be measured jointly if (and only if) the involved measurement accuracies $\Delta(A', A)$ and $\Delta(B', B)$ satisfy a relation, like

$$\Delta(A', A) \cdot \Delta(B', B) \geq c > 0, \tag{1}$$

where A' and B' are the marginal observables of an approximate joint observable M; $\quad A' = M_1, \; B' = M_2$.

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Again, such a relation exists for (Q, P) as well as for the qubit pairs.

How to define $\Delta(A', A)$?

The notions of error and disturbance are completely symmetric so that is suffice to consider e.g. the error.

Though there is no point in comparing individual measurement results of A and A′ in each case, one may compare the distributions A_ρ and A'_ρ in all input states.

The number $\Delta(A', A)$ should thus compare the distributions A_ρ and A'_ρ for all (or a relevant subset of) input states ρ with the obvious requirements:

$\Delta(\mathsf{A}', \mathsf{A}) = 0$
means that the "approximate" device A' is completely equivalent to the ideal A, i.e., for every input state ρ the output distributions A'_ρ and A_ρ will be the same.

$\Delta(\mathsf{A}', \mathsf{A}) < \varepsilon$
means that the difference in the distributions A'_ρ and A_ρ should also be small for every input state ρ.

Moreover, if the device only adds independent noise, that is, if all the distributions A'_ρ are smearings (convolutions) $\mathsf{A}'_\rho = \mu * \mathsf{A}_\rho$, of the distributions A_ρ with a fixed probability measure μ, then the error $\Delta(\mathsf{A}', \mathsf{A})$ should just give the 'size' of the noise, say, its standard deviation $\Delta(\mu)$.

Hence $\Delta(\mathsf{A}', \mathsf{A})$ **requires a 'distance' of probability measures**.

The Wasserstein distance (of order 2)

For $\Omega = \mathbb{R}$ we take $D(x,y) = |x-y|$.

- For a probability measure μ and a point measure δ_y define

$$
\begin{aligned}
D(\mu,\delta_y) &= \left(\int D(x,y)^2 \, d\mu(x)\right)^{\frac{1}{2}} = \left(\int |x-y|^2 \, d\mu(x),\right)^{\frac{1}{2}} \\
D(\mu) &= \inf_{y\in\mathbb{R}} D(\mu,\delta_y) = \inf_{y} \left(\int |x-y|^2 \, d\mu(x),\right)^{\frac{1}{2}}.
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\end{aligned}
$$

- For any two probability measures μ,ν, with a coupling $\gamma \in \Gamma(\mu,\nu)$,

$$
\begin{aligned}
D^{\gamma}(\mu,\nu) &= \left(\int D(x,y)^2\, d\gamma(x,y)\right)^{\frac{1}{2}} \\
D(\mu,\nu) &= \inf_{\gamma\in\Gamma(\mu,\nu)} D^{\gamma}(\mu,\nu)
\end{aligned}
$$

For any two observables A' and A and for any state ρ we have the distance $D(\mathsf{A}'_\rho, \mathsf{A}_\rho)$.

We take the worst case w.r.t ρ

$$D(\mathsf{A}', \mathsf{A}) = \sup_\rho D(\mathsf{A}'_\rho, \mathsf{A}_\rho)$$

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to represent the distance of the "approximator" A' from the "target" observable A.

We say that a bi- (or joint) observable M is an *approximate joint measurement* of A and B if the distances $D(\mathsf{M}_1, \mathsf{A})$ and $D(\mathsf{M}_2, \mathsf{B})$ are finite.

If A is sharp and, say, $\mathsf{M}_1 = \mu * \mathsf{A}$, i.e. $(\mathsf{M}_1)_\rho = \mu * \mathsf{A}_\rho$ for all ρ, for some μ, then

$$D(\mathsf{M}_1, \mathsf{A}) = D(\mu, \delta_0)$$

(For sharp observables we could also restrict to quantify over all "calibration states", that is states ρ in which A has a fairly sharp value x, that is, for given $\epsilon > 0$ the distance $D(\mathsf{A}_\rho, \delta_x) \leq \epsilon$ for some $x \in \mathbb{R}$. The results would be the same.)

Covariant phase space measurements

They are measurements (with two outcomes) which behave covariantly under phase space translations, that is, spatial translations and velocity boosts. Their structure is completely known.

The q- and p-marginals are the unsharp position and momentum with the Fourier related densities μ, ν:

$$\mathsf{M}^{Q} = \mathsf{M}_1 = \mu * \mathsf{Q}\,, \quad \mathsf{M}^{P} = \mathsf{M}_2 = \nu * \mathsf{P}$$

$$\mu = \mathsf{Q}_{\Pi\sigma\Pi}\,, \quad \nu = \mathsf{P}_{\Pi\sigma\Pi}\,, \quad \Pi \quad \text{the parity operator}$$

$$\sigma \quad \text{is the } \mathsf{M}\text{--defining density operator.}$$

A covariant phase space observable M^σ serves as an approximate joint measurement of Q and P:

$$\begin{aligned} D(\mathsf{M}^Q,\mathsf{Q})D(\mathsf{M}^P,\mathsf{P}) &= D(\mathsf{Q}_{\Pi\sigma\Pi},\delta_0)D(\mathsf{P}_{\Pi\sigma\Pi},\delta_0) \\ &\geq \Delta(\mathsf{Q}_{\Pi\sigma\Pi})\Delta(\mathsf{P}_{\Pi\sigma\Pi}) \geq \frac{1}{2}\hbar. \end{aligned}$$

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Note: Each M^σ has a quantum optical implementation as a high amplitude limit of the signal observable measured by an eight-port homodyne detector.

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Could one beat this result by another (phase space) measurement?

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Note: Each M^σ has a quantum optical implementation as a high amplitude limit of the signal observable measured by an eight-port homodyne detector.

Could one beat this result by another (phase space) measurement? No!

Theorem

Let M *be any (phase space) measurement which serves an approximate joint measurement of position and momentum, that is, the deviations* $D(\mathsf{M}_1, \mathsf{Q})$ *and* $D(\mathsf{M}_2, \mathsf{P})$ *are finite. Then*

$$D(\mathsf{M}_1, \mathsf{Q})D(\mathsf{M}_2, \mathsf{P}) \geq \frac{1}{2}\hbar.$$

The lower bound is obtained by an appropriate covariant measurement.

Similar results for qubit observables are obtained in the BLW paper *Heisenberg uncertainty for qubit measurements*, arXiv:1311.0837

What goes wrong in the NO approaches

The experiments which claim the refutation of the Heisenberg uncertainty relations rely on the *noise operator* (NO) based notions of error and disturbance.

In many cases (**noncommutative**) they are *operationally insignificant*, and even *wrong* measures of error and disturbance.

In those case where they are valid (**commutative**) they typically *overestimate* the (state dependent) error.

A detailed analysis with many examples is given in our forthcoming paper *Noise operators and measures of rms error and disturbance in quantum mechanics*.

The structure of these notions is the expectation of the squared difference of two operators, the ‘disturbed’ and ‘undisturbed’ ones, generically of the form

$$\epsilon_{\mathrm{NO}}(A', A, \rho \otimes \sigma)^2 = \langle (A' - A)^2 \rangle_{\rho \otimes \sigma}.$$

The quantum mechanical meaning of this number is the second moment of the statistics obtained when the observable (POM) defined by the difference operator $A' - A$ is measured in the state in question.

If the operators A' and A do not commute, a measurement of the difference observable has nothing to do with the measurements of its constituents.

Compare with: $H = T + V, \quad T = t(\mathrm{P}),\ V = v(\mathrm{Q})$.
One DOES NOT measure energy by measuring separately kinetic energy and potential energy and adding the results. There is NO VALUE of energy that would correspond to such a result.

Thanks!